Research Article
The Linear Complementarity Problem and a Method to Find all its Solutions

Y. El Foutayeni\textsuperscript{a}, H. El Bouanani\textsuperscript{b} and M. Khaladi\textsuperscript{c,d}

\textsuperscript{a}Analysis, Modeling and Simulation Laboratory, Hassan II University, Morocco
\textsuperscript{b}Department of Statistics and Applied Mathematics, Hassan II University, Morocco
\textsuperscript{c}Mathematical Populations Dynamics Laboratory, Cadi Ayyad University, Morocco
\textsuperscript{d}Unit for Mathematical and Computer Modeling of Complex Systems, IRD, France
UMI 209 UMMISCO, IRD - UPMC, France

Corresponding author: Y. El Foutayeni, e-mail: foutayeni@gmail.com

Received 15 May 2014; Accepted 20 June 2014

Abstract: To solve a linear complementarity problem LCP, almost all the authors assume that this problem admits one and only one solution, and they give a method to calculate this solution. The aim of this paper is to solve LCP when this problem has several solutions.

Keywords: Linear complementarity problem; Linear system; All solutions; Principal submatrix.

1 Introduction

The linear complementarity problem is to find a vector $x$ in $\mathbb{R}^n$ satisfying $x \geq 0$, $Ax + b \geq 0$ and $x^T(Ax + b) = 0$, or showing that no such vector exists, where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ are given data.

The linear complementarity problem, denoted by $LCP(A, b)$, introduced by Cottle \cite{3}, is one of the most widely studied mathematical programming problems. Solving $LCP(A, b)$ for an arbitrary matrix $A$ is NP-complete \cite{2}, while there are several classes of matrices $A$ for which the associated $LCP$s can be solved efficiently. For details of the theory of $LCP$s, see the books of Cottle et al. \cite{4} and Murty \cite{14}.

To solve the linear complementarity problem $LCP(A, b)$, almost all the authors assume that this problem admits one and only one solution (they impose certain conditions on the matrix $A$, in particular, positive-definite matrix, P-matrix, etc.), and they give an iterative method to calculate this solution (see \cite{1,5–13,15–18}, and many other papers), but the question is, when $LCP$ has several solutions, how to calculate them?
2 Preliminaries

The main result of this paper is to solve LCP where $A$ is an arbitrary matrix. In order to be able to do this, we first need some notation. We denote by $[n]$ the set of integers from 1 to $n$, and we will need the following definitions.

**Definition 2.1** Given $A \in \mathbb{IR}^{n \times n}$, let $X$, $Y$ be subsets of $[n]$. $A[X,Y]$ denotes the submatrix of $A$ having rows indexed by elements of $X$ and columns indexed by elements of $Y$.

**Definition 2.2** Given $b \in \mathbb{IR}^{n}$, let $X$ be a subset of $[n]$. $b[X]$ denotes the subvector of $b$ with rows indexed by elements of $X$.

We shall denote by $A[X]$ the principal submatrix $A[X,X]$, and we denote the complement of $X$ in the set $[n]$ by $\overline{X}$.

3 The Main Result

The key in our approach to finding all solutions to LCP is the following theorem.

**Theorem 3.1** For any matrix $A \in \mathbb{IR}^{n \times n}$ and vector $b \in \mathbb{IR}^{n}$, $LCP(A,b)$ has a solution if and only if there is a subset $X \subseteq [n]$ satisfying $A[X]x + b[X] = 0$ has a nonnegative solution $x^*[X] \geq 0$, and $A[X,X]x^*[X] + b[X] \geq 0$.

**Proof:** Let $x^*$ be a solution of the $LCP(A,b)$, that is: $x^* \geq 0$, $y^* = Ax^* + b \geq 0$ and $x^T y^* = 0$. Hence $x_i^* = 0$ or $y_i^* = 0$ for all $i \in [n]$. Now, let $X = \{ i \in [n] : x_i^* > 0 \}$ and let $\overline{X} = \{ i \in [n] : x_i^* = 0 \}$ the complement of $X$ in the set $[n]$. However, there are two possible cases. The first one: $X = \emptyset$, that is, $x^* = 0$ and $y^* = Ax^* + b = b$. This case is possible only if $b \geq 0$, then $x^* = 0$ is a solution of $LCP(A,b)$.

The second one: $X \neq \emptyset$, this means that, for all $i \in X$ we have $0 = y_i^* = \sum_{j \in X} a_{ij} x_j^* + b_i$ or in matrix form, $A[X]x^*[X] + b[X] = 0$; now for all $i \in \overline{X}$, we have $y_i^* = \sum_{j \in X} a_{ij} x_j^* + b_i$ or in matrix form $y^*[\overline{X}] = A[\overline{X},X]x^*[X] + b[\overline{X}]$. Thus, in both cases, there is a subset $X$ such that the system of linear equations $A[X]x + b[X] = 0$ has a nonnegative solution $x^*[X] \geq 0$ and $y^*[\overline{X}] = A[\overline{X},X]x^*[X] + b[\overline{X}] \geq 0$ with conventionally $x[\emptyset] = 0$.

Now we assume that there is a subset $X \subseteq [n]$ such that the system of linear equations $A[X]x + b[X] = 0$ has a nonnegative solution $x \geq 0$ and $y = A[\overline{X},X]x + b[\overline{X}] \geq 0$.

Let $x^* = (x_i^*)_{i \in [n]}$ and $y^* = (y_i^*)_{i \in [n]}$ where $x_i^* = \begin{cases} x_i & \text{if } i \in X \\ 0 & \text{if } i \in \overline{X} \end{cases}$ and $y_i^* = \begin{cases} y_i & \text{if } i \in X \\ y_i & \text{if } i \in \overline{X} \end{cases}$, it can be easily verified that $x^* \geq 0$, $y^* \geq 0$ and $x^T y^* = 0$ and therefore $x^*$ is a solution of the $LCP(A,b)$. This concludes the proof.

Note that if there is a unique subset $X \subseteq [n]$ satisfying (a) the matrix $A[X]$ is invertible; (b) $(A[X])^{-1}b[X] \leq 0$; (c) and $A[\overline{X},X](A[X])^{-1}b[X] - b[\overline{X}] \leq 0$. Then $LCP(A,b)$ has a unique solution.
Now, we denote the set of all subsets of \([n]\) by \(S_n = \{X \text{ such that } X \subseteq [n]\}\). It is known that the cardinality of the \(S_n\) is \(2^n\). We assume that the set \(S_n\) is sorted in increasing order (e.g., if \(n = 3\), then 
\(S_3 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}\)). Finally, we denote the set of solutions of \(LCP(A, b)\) by \(S\).

The next step is to give the following algorithm to compute the set \(S\).

**Algorithm**

For \(k = 0\) to \(2^n - 1\) do the following.

Let \(X_k \in S_n\). Recall that the cardinality of the empty set is 0.

Let \(x\) be a solution of the system of linear equations \(A[x_k]x + b[X_k] = 0\).

If \(x \geq 0\) and \(y[X_k] = A[X_k, x_k]x + b[X_k] \geq 0\), then \(x^* \in S\), where \(x^*[X_k] = x\) and \(x^*[X_k] = 0\);

else \(k \leftarrow k + 1\).

---

### 4 Numerical Example

To illustrate the result we will take the following simple example which admits four solutions.

Let us consider the following linear complementarity problem \(LCP\), find vector \(x\) satisfying \(x \geq 0\), \(Ax + b \geq 0\) and \(x^T(Ax + b) = 0\),

\[
A = \begin{bmatrix}
-3 & 1 & 2 \\
1 & 3 & 1 \\
2 & 1 & -3
\end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix}
-2 \\
3
\end{bmatrix}.
\]

The following table shows the different solutions of this problem. Column 1 and 2 show the step \(k\) and the subset \(X_k\), respectively; column 3 (resp. 4) reports the solution \(x\) to the system of equations \(A[X_k]x + b[X_k] = 0\) (resp. the vector \(y[X_k] = A[X_k, x_k]x + b[X_k]\)); column 5 presents the solution \(x^*\) to \(LCP\) (when it exists); and column 6 is the vector \(y^* = Ax^* + b\) (when it exists).

<table>
<thead>
<tr>
<th>(k)</th>
<th>(X_k)</th>
<th>(x^*[X_k])</th>
<th>(y^*[X_k])</th>
<th>(x^*)</th>
<th>(y^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\emptyset)</td>
<td>((0, 0, 0)^T)</td>
<td>((1, -2, 3)^T)</td>
<td>(*)</td>
<td>(*)</td>
</tr>
<tr>
<td>1</td>
<td>({1})</td>
<td>(\frac{1}{3})</td>
<td>((-\frac{5}{3}, \frac{11}{3})^T)</td>
<td>(*)</td>
<td>(*)</td>
</tr>
<tr>
<td>2</td>
<td>({2})</td>
<td>(\frac{2}{3})</td>
<td>((-\frac{5}{3}, \frac{11}{3})^T)</td>
<td>((0, \frac{2}{3}, 0)^T)</td>
<td>((\frac{5}{3}, 0, \frac{11}{3})^T)</td>
</tr>
<tr>
<td>3</td>
<td>({3})</td>
<td>1</td>
<td>((-1)^T)</td>
<td>(*)</td>
<td>(*)</td>
</tr>
<tr>
<td>4</td>
<td>({1, 2})</td>
<td>(\frac{1}{2}, \frac{1}{2})^T)</td>
<td>(\frac{9}{7})</td>
<td>(\frac{1}{2}, 0)^T)</td>
<td>((0, 0, \frac{9}{7})^T)</td>
</tr>
<tr>
<td>5</td>
<td>({1, 3})</td>
<td>(\frac{2}{3}, \frac{1}{3})^T)</td>
<td>(\frac{2}{3})</td>
<td>(0, \frac{11}{3})^T)</td>
<td>((0, 2, 0)^T)</td>
</tr>
<tr>
<td>6</td>
<td>({2, 3})</td>
<td>(\frac{3}{3}, \frac{11}{3})^T)</td>
<td>(\frac{11}{7})</td>
<td>(0, \frac{3}{3}, \frac{11}{3})^T)</td>
<td>((\frac{7}{2}, 0, 0)^T)</td>
</tr>
<tr>
<td>7</td>
<td>({1, 2, 3})</td>
<td>(\frac{3}{3}, \frac{2}{3}, \frac{11}{3})^T)</td>
<td>(*)</td>
<td>(*)</td>
<td>(*)</td>
</tr>
</tbody>
</table>

For \(k = 0, 1, 3, 7\), it is easy to show that in each case, whether the system \(A[X_k]x + b[X_k] = 0\) does not have a nonnegative solution \(x^*[X_k] \neq 0\), or \(y^*[X_k] = A[X_k, x_k]x^*[X_k] + b[X_k] \neq 0\). However, for \(k = 2, 4, 5, 6\), the system of linear equations \(A[X_k]x + b[X_k] = 0\) has a nonnegative solution \(x^*[X_k] \geq 0\), and \(y^*[X_k] = A[X_k, x_k]x^*[X_k] + b[X_k] \geq 0\).

Thus, \(S = \{(0, \frac{2}{3}, 0)^T, (\frac{1}{2}, 0, 0)^T, (\frac{3}{3}, 0, \frac{11}{3})^T, (0, \frac{3}{3}, \frac{11}{3})^T\}\).
References


Copyright 2014 Y. El Foutayeni, H. El Bouanani and M. Khaladi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.